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ON THE EDGE ACHROMATIC NUMBERS OF COMPLETE GRAPHS

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Suppose we wish to color the edges of the complete graph K_n with as many colors as possible so that (1) no two edges with a common node get the same color, and (2) for any two colors c_1 and c_2, there are two edges with a common node, one colored c_1 and the other colored c_2. What is the maximum number $A(n)$ of colors possible in such a coloring? Coloring problems are notoriously hard and this problem is no exception. In fact, a remarkable theorem of André Bouchet implies that an exact determination of $A(n)$ for all odd n would yield as a corollary all odd orders for which projective planes exist. Thus such a determination is clearly beyond the hopes of this study. The goals here are more modest: (1) to give a careful study of the best available upper bound on $A(n)$, (2) to add to the constructions which give reasonable lower bounds for $A(n)$, and (3) to contribute a few more values of n for which $A(n)$ is known exactly.

1. The achromatic index

Let G be a simple graph. A *proper* (*vertex*) *k-coloring* of G is a map of the vertices of G onto a set of k "colors" so that any two adjacent vertices of G receive different colors. Moreover, if for each pair of colors c_1 and c_2 there are adjacent vertices v_1 and v_2 so that v_i is colored c_i, then the coloring is *complete*. The smallest number k for which a coloring of G exists is the *chromatic number* $\chi(G)$ of G. Any coloring with $\chi(G)$ colors is necessarily complete since completeness means that it is impossible to merge any two color classes and still have a proper coloring. The largest k so that there exists a complete k-coloring of (the vertices of) G is the *achromatic number* $\psi(G)$ of G introduced by Harary and Hedetniemi [8]. An old result of Harary, Hedetniemi, and Prins [9] says that for any k between $\chi(G)$ and $\psi(G)$, a complete k-coloring of G exists. Thus the extreme values $\chi(G)$ and $\psi(G)$ determine the range of possible complete colorings. The achromatic number and the computational complexity of its determination have been studied by various authors. In general it appears that the exact determination of the achromatic number, even for simple structures such as trees, is quite difficult (cf. Lopez-Bracho [10] and Farber et al. [5]). However, an easy upper bound on $\psi(G)$ may be obtained as follows. If G has a complete k-coloring, then since there is an edge between each pair of color classes, G must have at least $|E| \geq k(k-1)/2$ edges. Hence $(k-1)^2 \leq k(k-1) \leq$

$2|E|$, so

$$\psi(G) \leq \sqrt{2|E|} + 1 \tag{1.1}$$

The purpose here is to investigate the achromatic number $A(n)$ of the line graph $L(K_n)$ of the complete graph K_n. Notationally, $A(n) = \psi(L(K_n))$. As is well-known, the chromatic index $\chi'(L(K_n))$ is n if n is odd and $n-1$ if n is even. That the precise determination of the achromatic indices $\psi(L(K_n))$ will be much more difficult is evident from a remarkable result of Bouchet [3] stated below. A complete edge-coloring of K_n with the maximum number $A(n)$ of colors will be called an *optimal coloring*. If Γ is any color class in an edge-coloring, then the nodes covered by the edges in Γ will be called the *support of* Γ.

Theorem 1.2 (Bouchet). *Suppose q is odd and $n = q^2 + q + 1$. Then $A(n) = qn$ if and only if a projective plane of order q exists. Indeed, if $A(n) = qn$, then the supports of the color classes in any optimal coloring form the lines of a projective plane with the nodes of K_n as points.*

Aside from the values given by Bouchet's theorem, the exact value of $A(n)$ is now known only for $n \leq 11$ and $n = 25$. The best current estimates on $A(n)$ for $n \leq 100$ are summarized in the last section.

In general, since $L(K_n)$ is regular of degree $2(n-2)$, inequality (1.1) yields the following upper bound:

$$A(n) \leq \sqrt{n(n-1)(n-2)} + 1 \leq (n-1)^{\frac{3}{2}} + 1. \tag{1.3}$$

Although this bound can be improved slightly, it is of the right order of magnitude. The proof uses the monotonicity $A(n+1) \geq A(n)$ which is a consequence of the following simple lemma.

Lemma 1.4. *For any graph G, if H is an induced subgraph of G, then $\psi(G) \geq \psi(H)$.*

Proof. It suffices to show this if H is obtained from G by deleting a single node v. Given a complete coloring C of H, extend this to a complete coloring of G by either (1) coloring v with a new color if all the colors of C appear on neighbors of v in G, or (2) coloring v with a color of C not on any neighbor of v otherwise. □

Theorem 1.5. $A(n)/n^{\frac{3}{2}} \to 1$ *as* $n \to \infty$.

Proof. The proof depends on a strengthened version, due to Tchebychev, of Bertrand's "Postulate" which follows from the Prime Number Theorem (cf. Gioia [6]): For any $\varepsilon > 0$, there is an N_ε such that for any real $x \geq N_\varepsilon$, there is a prime p between x and $(1+\varepsilon)x$. Now let $\varepsilon > 0$ be given, and suppose $n > (N_\varepsilon + 1)^2(1+$

$\varepsilon)^2$. Set $x=[\sqrt{n}/(1+\varepsilon)]-1$, so $x \geqslant N_\varepsilon$. We may then select a prime p with $x \leqslant p \leqslant (1+\varepsilon)x$. Note that $p^2+p+1<(x+1)^2(1+\varepsilon)^2=n$. Since projective planes of all prime orders exist (cf. Hall [7]), it follows from Bouchet's theorem and Lemma 1.4 that

$$A(n) \geqslant A(p^2+p+1)=p(p^2+p+1)>p^3 \geqslant x^3=(\sqrt{n}-1-\varepsilon)^3/(1+\varepsilon)^3.$$

Since ε was arbitrarily small, the result follows. □

Since $A(n)$ grows asymptotically like $n^{\frac{3}{2}}$, one might expect to have $A(n+1)-A(n)=O(\sqrt{n})$, but this remains unproved. The best known result on the difference $A(n+1)-A(n)$ is the trivial inequality $A(n) \geqslant A(n+1)-n$ obtained by deleting any vertex and the n incident color classes from any optimal coloring of K_{n+1}. It is also characteristic of the quirks of this problem that no proof of the *strict* inequality $A(n)<A(n+1)$ is known in general, although this is almost surely the case. The constructions given in Section 3 do confirm this strict inequality for an infinite class of n, and the result below establishes a two-step strict monotonicity.

Theorem 1.6. *$A(n+2) \geqslant A(n)+2$ if $n>4$.*

Proof. Consider an optimal coloring of K_n. Select a maximal collection Γ of disjoint edges of different colors, which implies that Γ meets every color class. Let st be an edge of Γ. The $n-2$ other edges at t all have distinct colors. Since Γ is a matching, it contains at most $n/2$ edges. Hence there is an edge tu whose color does not occur in Γ. Starting with tu, select a maximal collection Δ of disjoint edges colored with colors not used for Γ. The subgraph G generated by $\Gamma \cup \Delta$ is bipartite because the bipartition $\{\Gamma, \Delta\}$ is a proper 2-coloring of its edge set. So the vertices of G may be properly colored black and white.

Now add two new nodes b and w. Let xy be an edge of G where x is black and y is white. If $xy \in \Gamma$, color the edges bx and wy both with the color of xy. If $xy \in \Delta$, color the edges wx and wf both with the color of xy. Since there are at most two edges (one from Γ and one from Δ) at each node of G, this is a consistent coloring. Moreover, since all the colors in $\Gamma \cup \Delta$ are different, no color appears twice at either b or w. Now erase the old colors on the edges of Γ and make Γ a new color class. Notice for any edge xy of Γ, its old color class still has x and y as well as b and w in its support in the new coloring. Now erase the old colors on the edges of Δ and make $\Delta^*=\Delta \cup \{bw\}$ a new color class. This is a proper coloring by the remarks above.

Since the supports of old colors are either left the same or enlarged by b and w, it follows that any two old color classes still meet. Moreover, Γ meets every old color and Δ^* meets every old color not originally on an edge of Γ. But edge bw meets all these colors. Finally, Γ and Δ^* meet on the special edges st and tu. Hence the coloring is complete. □

From the computational viewpoint, Yannakakis and Gavril [14] showed the following problem to be NP-complete: Given a graph G and an integer n, is $\psi(G) \geqslant n$? But for fixed n, Farber et al. [5] proved there is an algorithm which, for an abritrary graph G, determines in $O(|E(G)|)$ time whether $\psi(G) \geqslant n$. Their proof was nonconstructive and they were able to exhibit such an algorithm only for $n \leqslant 4$. Since by Bouchet's theorem $\psi(L(K_{241})) \geqslant 3615$ is equivalent to the existence of a projective plane of order 15, one may expect that the constant involved is quite large.

2. Upper bounds

In this section a refinement of the bound in (1.3) is presented and studied. Except for a few values where an improvement by 1 is possible, the result is the best upper bound known for $A(n)$.

The following functions play a crucial role:

$$g(x, y) = 2y(x - y - 1) \quad \text{and} \quad h(x, y) = x(x-1)/(2y). \tag{2.1}$$

Note (say, by differentiation) that for fixed x, the quadratic $g(x, y)$ in y is increasing for $y < (x-1)/2$.

Lemma 2.2. *For any $t < (n-1)/2$, $A(n) \leqslant \max\{g(n, t) + 1, h(n, t+1)\}$.*

Proof. Consider any complete k-coloring of $L(K_n)$. Suppose first that there is a color class Γ with exactly $s \leqslant t$ edges of K_n in it. Let S be the set of $2s$ nodes of K_n covered by the s edges in Γ. An edge of K_n is adjacent to an edge of Γ in $L(K_n)$ iff it has an endnode in S. There are $n-1$ edges of K_n incident with each point of S; there are $s(2s-1)$ edges of K_n incident with two points of S. Hence the number of edges of K_n not in Γ but incident with a point of S is

$$2s(n-1) - s(2s-1) - s = g(n, s).$$

Since Γ must be adjacent to at least one edge of every other color class, it follows that

$$k \leqslant g(n, s) + 1 \leqslant g(n, t) + 1.$$

Now if no color class contains t or fewer edges, then every color class contains at least $t+1$ edges. In this case, the number of color classes is at most $h(n, t+1)$, and the lemma is proved. $\square$

We now wish to give an explicit description of the bound implied by Lemma 2.2. Toward this end, let

$$\beta_t(n) = \max\{g(n, t) + 1, \lfloor h(n, t+1) \rfloor\}$$

where $\lfloor x \rfloor$ denotes the greatest integer in x. Now set

$$B(n) = \min\{\beta_t(n): 0 < t < (n-1)/2\}$$

Then Lemma 2.2 may be reformulated succinctly as $A(n) \leqslant B(n)$. This represents the best known upper bound on $A(n)$, except for a sparse set of values (discussed below) where the bound can be improved to $B(n) - 1$. It is therefore convenient to have a direct rather than min-max description of $B(n)$. With t fixed, $g(n, t)$ grows linearly in n and $h(n, t+1)$ grows quadratically. Initially, g is in the lead but at some point, h takes over. Eventually, $h(n, t+1)$ outgrows even $g(n, t+1)$ and the value of t minimizing β_t switches from t to $t+1$, and the scenario repeats. The exact crossover values are described in the following technical lemma.

Lemma 2.3 *Suppose $t \geqslant 2$. If $4t^2 - t \leqslant n \leqslant 4t^2 + 3t - 1$, then $B(n) = g(n, t) + 1$. If $4t^2 + 3t \leqslant n \leqslant 4(t+1)^2 - t - 2$, then $B(n) = \lfloor h(n, t+1) \rfloor$.*

Proof. We need to compare g with $\lfloor h \rfloor$. Notice that since g is integral, $g(n, t) + 1 \leqslant \lfloor h(n, t) \rfloor$ iff $g(n, t) + 1 \leqslant h(n,t)$. By subtraction, this is equivalent to $0 \leqslant h(n, t) - g(n, t) - 1$. Multiplied through by $2t$, the right side becomes a polynomial in n and t:

$$p(n, t) = n^2 - (4t^2 + 1)n + 4t^3 + 4t^2 - 2t.$$

Thus $g(n, t) + 1 \leqslant \lfloor h(n, t) \rfloor$ iff $p(n, t) \geqslant 0$. Similarly, $g(n, t) + 1 \leqslant \lfloor h(n, t+1) \rfloor$ iff $q(n, t) \geqslant 0$ where

$$q(n, t) = n^2 - (4t^2 + 4t + 1)n + 4t(t+1)^2 - 2(t+1).$$

Tedious but routine evaluations reveal:

$$\begin{aligned} p(4t^2 - t - 1, t) &= -3t^2 + t + 1 < 0 && \text{if } t > 0 \\ p(4t^2 - t, t) &= t^2 - t \geqslant 0 && \text{if } t > 0 \\ q(4t^2 + 3t - 1, t) &= -3t^2 - 3t < 0 && \text{if } t > 0 \\ q(4t^2 + 3t, t) &= t^2 - t - 2 \geqslant 0 && \text{if } t \geqslant 2 \end{aligned} \tag{2.4}$$

Now let $t \geqslant 2$ be fixed.

First let us investigate the range between $4t^2 - t$ and $4t^2 + 3t - 1$. Differentiating, we find $D_x q(x, t) = 2x - (4t^2 + 4t + 1)$. For $x \geqslant 4t^2 - t$, this is positive, so $q(x, t)$ is increasing. Since $q(4t^2 + 3t - 1, t) < 0$ by (2.4) it follows that $q(n, t) < 0$ for all n in the range $4t^2 - t \leqslant n \leqslant 4t^2 + 3t - 1$. Thus for such an n, $h(n, t+1) < g(n, t) + 1$, so $\beta_t(n) = g(n, t) + 1$.

Now if $t < u < (n-1)/2$, then $g(n, u) + 1 \geqslant g(n, t) + 1 > h(n, t+1) \geqslant h(n, u + 1)$ since $g(x, y)$ is increasing in y (for $y < (x-1)/2$) and $h(x, y)$ is decreasing in y. It follows that $\beta_u(n) \geqslant \beta_t(n)$.

Now consider $s < t$. Differentiating, we find $D_x p(x, t) = 2x - (4t^2 + 1)$. For $x \geqslant 4t^2 - t$, this is positive, so $p(x, t)$ is increasing. Since $p(4t^2 - t, t) \geqslant 0$ by (2.4), it follows that $p(n, t) \geqslant 0$ for all $n \geqslant 4t^2 - t$. Thus for such an n, $\lfloor h(n, t) \rfloor \geqslant g(n, t) + 1$. Hence for $s < t$, we have $\beta_s(n) \geqslant \lfloor h(n, s+1) \rfloor \geqslant \lfloor h(n, t) \rfloor \geqslant g(n, t) + 1 = \beta_t(n)$. It follows that $B(n) = \beta_t(n) = g(n, t) + 1$ as desired.

Now let us consider the range from $4t^2 + 3t$ to $4(t+1)^2 - t - 2$. As noted above, $q(x, t)$ is increasing for $x \geqslant 4t^2 - t$. Since $q(4t^2 + 3t, t) > 0$ by (2.4), it follows that $q(n, t) > 0$ and hence $\lfloor h(n, t+1) \rfloor \geqslant g(n, t) + 1$ for $n \geqslant 4t^2 + 3t$. Thus for such n, we have $\beta_t(n) = \lfloor h(n, t+1) \rfloor$.

Now if $s < t$, then $\beta_s(n) \geqslant \lfloor h(n, s+1) \rfloor \geqslant \lfloor h(n, t+1) \rfloor = \beta_t(n)$. To finish the proof, it suffices to show $\beta_u(n) \geqslant \beta_t(n)$ for $t + 1 \leqslant u < (n-1)/2$. To this end, consider the derivative $D_x p(x, t+1) = 2x - 4(t+1)^2 - 1$ which is positive if $x \geqslant 4t^2 + 3t$. Since $p(4(t+1)^2 - (t+1) - 1, t+1) < 0$ by (2.4), it follows that $p(n, t+1) < 0$ for all n in the range $4t^2 + 3t \leqslant n \leqslant 4(t+1)^2 - t - 2$. Hence for n in this range, we have $\beta_t(n) = \lfloor h(n, t+1) \rfloor \leqslant g(n, t+1) + 1 \leqslant g(n, u) + 1 \leqslant \beta_u(n)$. Thus $B(n) = \beta_t(n) = \lfloor h(n, t+1) \rfloor$ as desired. □

Theorem 2.5 *If $t > 1$ and $n = 4t^2 - t$, then $A(n) \leqslant B(n) - 1$.*

Proof. Let $n = 4t^2 - t$, and suppose there is a complete coloring of $L(K_n)$ with $B(n)$ colors. By Lemma 2.3, $B(n) = g(n, t) + 1$ in this case. If there were a color class with $s < t$ colors, it would meet only $g(n, s) < g(n, t)$ edges and hence could not meet all the other color classes. Thus every color class has at least t edges. Moreover, any color class Γ with exactly t edges meets exactly $g(n, t)$ other edges, so these must all have different colors in order for Γ to meet all the other color classes. The number of color classes with more than t edges is at most the total number of edges in K_n minus t times the number of color classes. A tedious but crucial calculation reveals this to be $n(n-1)/2 - t(g(n, t) + 1) = (t^2 + t)/2$. Now pick a color class Γ with exactly t edges. There are $2t$ nodes in the support of Γ and hence $(2t(2t-1)/2) - t = 2(t^2 - t)$ edges not in Γ which pass between these nodes. As noted above, all these edges must have distinct colors, so since $2(t^2 - t) > (t^2 + t)/2$, at least one of these edges, say ab, must come from a class Δ with exactly t edges. Let au and bv be the edges in Γ incident with ab. But then au and bv are distinct edges having the same color and incident with the t-edge class Δ. This contradicts the fact that all edges incident with Δ must have different colors. □

Bouchet's Theorem in conjunction with the Bruck–Ryser Theorem (cf. Hall [7], p. 175) also yields (a much deeper!) improvement of the bound $B(n)$ for certain n. Indeed, if q is odd and $n = q^2 + q + 1$, then setting $t = (q+1)/2$ it is easy to see that $B(n) = h(n, t) = qn$. Thus we have

Theorem 2.6 (Bouchet, Bruck–Ryser). *Suppose $q \equiv 1 \bmod 4$ and $n = q^2 + q + 1$. If q is not a sum of squares, then $A(n) \leqslant B(n) - 1$.*

There remains one other case in which it is known that $A(n) \leq B(n) - 1$. One can easily verify that $B(6) = 9$. However, Bories [2] and later independently Turner [12] both realized that $A(6) = 8$. As this result was not previously published, it is given here for completeness.

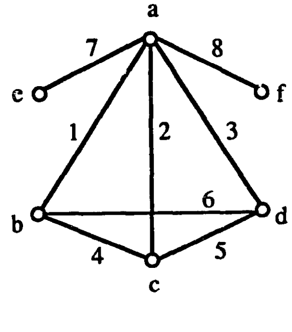
Theorem 2.7 (Bories, Turner). $A(6) \leq 8$.

Proof. Suppose on the contrary that $L(K_6)$ has a complete 9-coloring. For convenience, call an edge which forms a color class by itself a *singleton edge*. If there were at most two singleton edges, then in all there would be at least $2(7) + 2 = 16$ edges, a contradiction. Hence there are at least three singleton edges. Since an edge meets exactly 8 other edges, it follows that the 8 edges incident with a singleton edge must all receive different colors. Moreover, the singleton edges must meet each other. Hence they form either a star or a triangle.

Suppose first that the singleton edges are *ab, ac, ad* and that they are colored 1, 2, 3. Since *bc, cd, db* form a triangle, they must receive distinct colors. Since colors 1, 2, 3 are used just once, say that *bc, cd, db* are colored 4, 5, 6, respectively. Let *e* and *f* be the remaining vertices of K_6. Now *ae* cannot be colored 4 or 6 since *ab* is already adjacent to these colors. Similarly, *ae* cannot be colored 5 since *ac* is already adjacent to color 5. Hence *ae* (and similarly *af*) must be colored 7 or 8. Say, *ae* receives 7 and *af* receives 8. (See Fig. 1a).

Note that *ab* is now adjacent with all colors *except* 5 and 9. Thus the colors assigned *be* and *bf* must be 5 and 9, or vice versa. Similarly, *ce* and *cf* must receive 6 and 9, and *de* and *df* receive 4 and 9. But this forces two edges colored 9 to be incident with one of the vertices *e* or *f*, a contradiction.

Now suppose the singleton edges are *ab, bc, ca* and that they are again colored 1, 2, 3. Since all edges incident with *ab* receive different colors, we may assume the coloring is as shown in Fig. 1b. The as yet uncolored edges adjacent to *ac* are



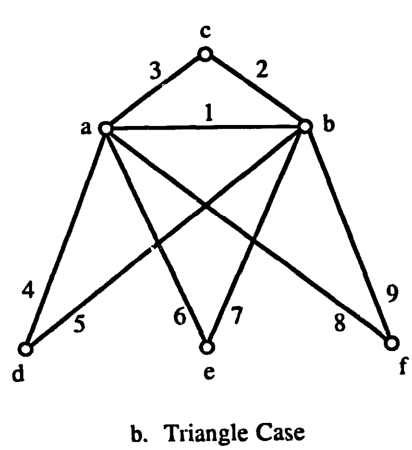


Fig. 1. Impossibility of 9-coloring K_6. a. Star case. b. Triangle case.

cd, ce, cf and these must be colored 5, 7, 9 (in some order) so that *ac* is adjacent to all colors. Similarly, the same edges *cd, ce, cf* are the uncolored edges adjacent to *bc*, and these must be colored 4, 6, 8, a contradiction. □

3. Constructions from projective planes

Suppose q is the order of a projective plane P. Then P has $n = q^2 + q + 1$ points and n lines. We may regard the points of P as the nodes of a complete graph K_n. Each edge lies in a unique line which is a copy of K_{q+1}. Now if q is *odd*, then K_{q+1} has a 1-factorization. Each line may thus be divided into q color classes, each of size $(q+1)/2$. All colors of a line are incident with all points of that line. Hence since any two lines meet, the coloring is complete. This yields the easy part of Bouchet's Theorem: $A(n) = qn$ if q is odd and the order of a projective plane.

If q is even, then K_{q+1} is not 1-factorable, so the argument collapses. However, by adjoining some additional points, it is possible to obtain some good (if not exact) lower bounds.

Theorem 3.1. *Suppose q is even and the order of a projective plane. Let $n = q^2 + 2q + 2$. Then $A(n) \geqslant nq + 1$ and $A(n+1) \geqslant nq + q + 2$.*

Proof. Let P be a projective plane of order q regarded as a complete graph. Each line L of P has an odd number $q + 1$ of points. Hence L may be edge-colored with $q + 1$ colors. In any such coloring, at each point v of L, there will be exactly one color missing. Moreover, since each color class in L must contain $q/2$ edges, different points of L will be missing different colors in L.

Using a different set of $q + 1$ colors for each line L of P, we obtain an edge-coloring of P with $(q+1)(q^2+q+1) = qn + 1$ colors. It is a proper coloring, but because of the "missing" colors, it fails to be complete.

For each incident point-line pair (v, L) in P, let $c(v, L)$ denote the missing color at v in the line L. These colors are all different and thus comprise the full set of $qn + 1$ colors. Now arbitrarily order the lines through each point v as $L_0(v), L_1(v), \ldots, L_q(v)$. Let $U = \{u_0, u_1, \ldots, u_q\}$ be a set of $q + 1$ new points. Then K_n may be represented with the n points of $P \cup U$ as node set. Color each edge vu_i with the missing color $c(v, L_i(v))$. Since the colors $c(v, L)$ are all distinct, this remains a proper coloring. Now for each $v \in L$, we have all colors of L occurring on edges at v. Since any two lines of P have a point in common, it follows that any two color classes are incident, so the coloring is complete.

The edges between the u_i's have not yet been colored. However, we do have a complete $(qn+1)$-coloring of a subgraph of $L(K_n)$. Hence by Lemma 1.4, we have $A(n) \geqslant qn + 1$.

Now continue the above construction by adding another new point u^*. Set

$U^* = U \cup \{u^*\}$. Then U^* has an *even* number $q+2$ of points and hence is 1-factorable. That is, U^* can be edge-colored with $q+1$ colors, each color occurring at each point of U^*. Since all previous $qn+1$ colors are incident at the points of U, it follows that we may choose these $q+1$ colors to be new and still have a complete coloring. Thus $A(n+1) \geq nq+q+2$. □

With n as above, taking $t = q/2$ in Lemma 2.3, we see that $B(n-1) = nq - q$, $B(n) = nq + q$ and $B(n+1) = nq + 3q + \lfloor 6/(q+2) \rfloor$. Thus by the above constructions, we have $A(n-1) \leq B(n-1) < A(n) \leq B(n) < A(n+1)$, so the strict monotonicity of A is established for infinitely many values of n.

Theorem 3.2. *Let q be a power of 2, and let s be an integer such that $\frac{1}{2}q + 1 \leq s \leq q$. Then $A(q^2+q+s) \geq q^2(q+1) + \min\{(2s-q-1)(q+1), A(s)\}$.*

Proof. As in the above construction, let P be a projective plane of order q and edge-color each line of P with $q+1$ colors. Now fix a point p of P and denote the lines of P through p by $M_0, M_1, \ldots, M_q$. (We will call these *M-lines* and the lines not through p will be called *L-lines*.) Arbitrarily order the points $\neq p$ on M_k as $w(k, j)$, $j = 1, \ldots, q$. Now delete p and remove all colors from the M-lines. This leaves $q^2(q+1)$ "old" colors.

For each point $v \neq p$, order the lines through v as $L_0(v), L_1(v), \ldots, L_q(v)$ where $L_0(v)$ is the M-line through v and p. Let $U = \{u_1, \ldots, u_s\}$ be a set of new points. For each $v \neq p$ and i with $1 \leq i \leq s$, color edge vu_i with the "missing" color $c(v, L_i(v))$ at v on line $L_i(v)$. There then remain $q-s$ "missing" colors at each point v.

Say, $v = w(k, j)$. For each i with $s < i \leq q$, color the edge from $w(k, j)$ to $w(k, j+i-1)$ with the color $c(v, L_i(v))$. (The sum $j+i-1$ is modulo q.) Now every color on an L-line occurs at each point of that L-line. Since any two L-lines intersect at a point $\neq p$, the coloring so far is complete. Morever, the edges in U and certain edges on the M-lines are still available to be colored.

Consider some M_k and some i with $\frac{1}{2}q + 1 \leq i \leq s$. The edges $w(k, j)$ to $w(k, j+i-1)$ on M_k are as yet uncolored. If $i = \frac{1}{2}q + 1$, these edges from a 1-factor. Otherwise, these edges form a collection of cycles of length λ where λ is the least integer such that q divides $\lambda(i-1)$. Since q is a power of 2, λ must be even so this collection of edges may be split into two 1-factors. Thus on each M_k, we have available $2(s - \frac{1}{2}q - 1) + 1$ matchings, each of which covers all points of M_k. Since each L-line meets each M-line at a point $\neq p$, it follows that if one of these matchings is taken as a new color class, then it will meet all old colors. Therefore the difficulty in adding new colors lies only in making new color classes from different M-lines meet each other. This can be accomplished by repeating the new colors in a complete edge-coloring of U. □

Suppose we fix a constant $c \geq 1$ and take $s = \frac{1}{2}q + \gamma$ where $1 \leq \gamma \leq c$. As q

Table 1. A residual edge-coloring of K_{24}.

M_0:	A	A'	(D, D')
M_1:	B	B'	(A, A')
M_2:	C	C'	(B, B')
M_3:	D	D'	(C, C')
M_4:	E	(B, B')	(D, D')
U:	E	(A, A')	(C, C')

grows, $A(s)$ grows like a constant times $q^{\frac{3}{2}}$ by (1.5), so $A(s)$ eventually dominates $(2s-q-1)(q+1)=(2\gamma-1)(q+1)$. Hence the above result yields $A(q^2+\frac{3}{2}q+\gamma)\geqslant q^3+q^2+(2\gamma-1)(q+1)$ for large q. This compares favorably with the upper bound B—in fact, well enough to imply strict monotonicity in the range under consideration.

It is possible to improve the bound in Theorem 3.2 by taking advantage of the fact that new colors on the same M-line already meet each other. This is illustrated by the following special constructions.

Theorem 3.3. $A(24)\geqslant 89$ *and* $A(78)\geqslant 591$.

Proof. First, take $s=q=4$ and proceed with the construction in (3.2) up to the place where the new colors are to be chosen. The edges to receive new colors form six disjoint K_4's—the five M-lines and U. Each K_4 factors into three pairs of edges. Table 1 shows a method of coloring these edges with 9 new colors A, A', B, B', C, C', D, D', and E. A new color standing alone indicates that both edges of a 1-factor are to receive that color. A pair of colors in parentheses indicates that the listed colors are to be distributed over the edges of a 1-factor. Since each new color stands alone on some M-line (and hence has that M-line in its support), each new color meets every old color. It is easily verified that the new colors are pairwise incident.

In the same spirit, Table 2 shows a scheme for the new colors in the case $q=8$, $s=6$. Graphically, the edges to be colored on each M-line form an 8-cycle with

Table 2. A residual edge-coloring of K_{78}.

M_0:	A A'	(B, B', C, C')
M_1:	B B'	(C, C', C, D')
M_2:	C C'	(D, D', E, E')
M_3:	D D'	(E, E', F, F')
M_4:	E E'	(F, F', A, A')
M_5:	F F'	(A, A', B, B')
M_6:	G	$(A, D, A', D', A, C, F, C')$
M_7:	H	$(B, E, B', E', B, C, F', C')$
M_8:	I	$(C, C', D, D', E, E', F, F')$

	u_1	u_2	u_3	u_4	u_5	u_6
u_1	*	G	H	E	E'	F
u_2	G	*	I	B	B'	F'
u_3	H	I	*	A	A'	D
u_4	E	B	A	*	H	D'
u_5	E'	B'	A'	H	*	I
u_6	F	F'	D	D'	I	*
			U			

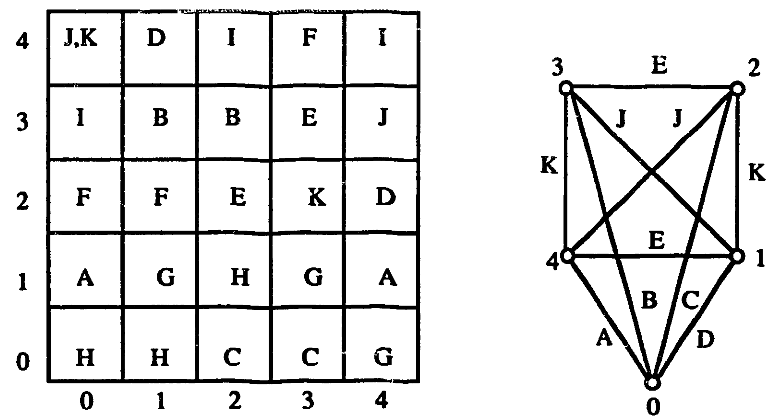


Fig. 2. A residual edge-coloring of K_{30}.

its four principal diagonals. For M_0 through M_5, imagine the edges split into three 1-factors, each of the first two receiving a single color and the third receiving four different colors as indicated. On M_6, M_7, and M_8, the diagonals receive colors G, H, and I, respectively, and the 8-cycles are colored as indicated. Finally, a special color table is given for the edges of U. The verification that the resulting coloring is proper and complete is easy and left to the reader. □

To conclude this section, we give two special constructions based on removing a point from an odd order projective plane

Theorem 3.4 *$A(12) \geqslant 31$ and $A(30) \geqslant 136$.*

Proof. The projective plane PG(2, 5) over GF(5) may be regarded as consisting of

(1) the 25 points of the affine plane AG(2, 5)—represented in Fig. 2 by the 5×5 square grid,
(2) the 5 points on the line at infinity corresponding to "slopes" of nonvertical lines—represented in Fig. 2 by the vertices of K_5, the vertex labels being slopes, and
(3) one "vertical" point on the line at infinity.

Take an optimal coloring of K_{31} based on the plane PG(2, 5). This has $qn = 155$ colors. Remove the vertical point, the 30 edges and the 30 colors incident with it. This leaves 30 points with 125 "old" colors intact and 60 edges which must be recolored. Geometrically, the uncolored edges lie on the 5 vertical lines of AG(2, 5) and the "punctured" line at infinity. Graph theoretically they from 6 disjoint copies of K_5. Fig. 2 provides a scheme for recoloring these edges with 11 "new" colors $A, B, \ldots, J, K$.

The colors of the 10 edges on the infinite line are as indicated. The colors on the other vertical lines are assigned as follows. Color "X" in grid position (i, j)

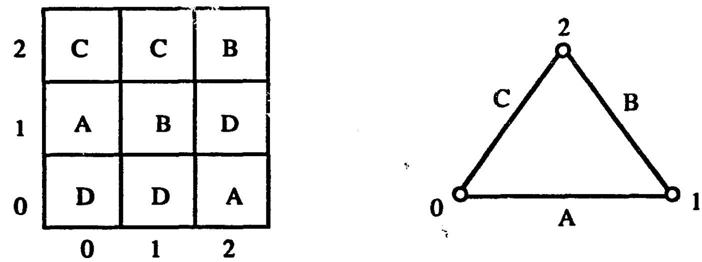


Fig. 3. A residual edge-coloring of K_{12}.

means: on the vertical line $x = i$, color edges $(i, j-1)$, $(i, j+1)$ and $(i, j-2)$, $(i, j+2)$ with color X. The ambiguity in the $(0, 4)$ position is to be resolved by coloring edge $(0, 3)$, $(0, 0)$ with J and edge $(0, 2)$, $(0, 1)$ with K.

The result is an edge coloring of K_{30} with 136 colors, 125 old and 11 new. Let us check that it is proper. First note that the coloring on the infinite line is a complete (in fact, optimal) edge coloring of K_5 with 7 colors. That the coloring is proper on the remaining 5 vertical lines follows from the fact that no color appears more than once in any column of the grid representing AG(2, 5).

The check that this coloring is complete is more tedious and involves verifying that the support of each new color meets every nonvertical line. The details, which are routine, are left to the reader.

The data in Fig. 3 may be used in a similar way to define a complete 31 coloring of the edges of K_{12}. □

The same procedure may be applied to the coloring induced by any odd order plane. But as the order grows, the recoloring becomes every more tedious. Moreover, the requirement that the new colors "block" all the nonvertical lines, together with results of Aiden Bruen [1] on blocking sets, suggests that the new color classes must be too large to yield an efficient coloring in general.

4. Group divisible colorings

In this section we shall further exploit a construction technique introduced by Bouchet [3]. Let G be a group of order n. A *Bouchet diagram* (over G) is a (simple) graph D such that

(1) D has a 1-factorization,
(2) the nodes of D are elements of G,
(3) for any edges $[x, y]$ and $[u, v]$ of D, $x^{-1}y = u^{-1}v$ if and only if $x = u$ and $y = v$,
(4) for any g in G, there are nodes x and y of D with $xy^{-1} = g$.

If D has m nodes and is regular of degree d, then we shall call it an

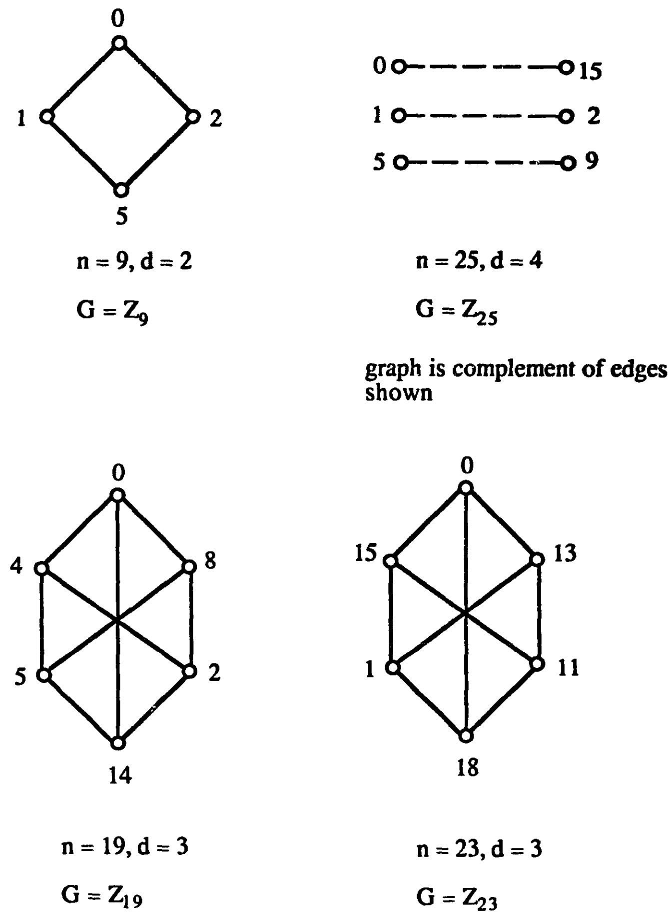


Fig. 4. Bouchet diagrams.

(n, m, d)-Bouchet diagram. Let $D^* = \{x^{-1}y\colon x$ and y are adjacent in $D\}$. Note that $D^* = (D^*)^{-1}$ since adjacency is symmetric, that D^* does not contain the identity since D is loopless, and D^* does not contain any involution by (3). It is often convenient to think of D as a labelled graph: by (2) the nodes of D are labelled by elements of G, and each edge $[x, y]$ is labelled by a pair of inverse elements $\{x^{-1}y, y^{-1}x\}$. Fig. 4 shows four Bouchet diagrams. For $n = 9$, 19 these are due to Bouchet [3]; for $n = 23$, 25 they are new.

The *Cayley graph* Cay(G, D^*) has the elements of G as nodes and an edge $[g, h]$ whenever $g^{-1}h$ is in D^*. The diagram D induces a complete edge coloring of Cay(G, D^*) as follows. For each $g \in G$, let gD denote the translate of D by g—that is, the nodes of gD are of the form gx where $x \in D$ and two nodes gx and gy are adjacent in gD iff x and y are adjacent in D. The translates then cover the edges of Cay(G, D^*). Condition (3) guarantees that the translates gD are edge disjoint and condition (4) guarantees that any two translates of D have at least one node in common. Select a 1-factorization of each gD and regard this as an edge-coloring with d colors of gD, using disjoint sets of colors for different translates. Since all colors used in any gD occur at all nodes of gD and since any two translates have a node in common, it follows that this coloring is complete. Since dn colors are used and since Cay(G, D^*) is an edge-subgraph of the complete graph on G, we have the following result.

Theorem 4.1 (Bouchet). *If an (n, m, d)-Bouchet diagram exists, than $A(n) \geq dn$.*

For $n = 9$ and $n = 25$, the colorings generated by the diagrams in Fig. 2 are optimal since $B(9) = 18$ and $B(25) = 100$. It is possible that these may be the first in a series for n an odd square but no general constructions are known yet. For $n = 19$, the coloring is maximal among all colorings with class sizes 3 or more. But it may be possible to obtain more colors by allowing some color classes of two edges. For $n = 23$, a better coloring was obtained using projective planes (Theorem 3.2 with $q = 4$ and $s = 3$).

The following result provides a method for augmenting a group divisible coloring.

Theorem 4.2. *If an (n, m, d)-Bouchet diagram exists, then for all k, $A(n + km) \geq (d + k)n$.*

Proof. As before, take the induced coloring on Cay(G, D^*). Introduce km new points $u_i(x)$ where $i = 1, \ldots, k$ and $x \in D$, and introduce kn new colors $c_i(g)$ where $i = 1, \ldots, k$ and $g \in G$. Now color the edge $[g, u_i(x)]$ with color $c_i(gx^{-1})$. For fixed i and x, as g ranges through G, the products gx^{-1}, and hence the colors $c_i(gx^{-1})$, are all distinct. Similarly, when g is fixed and x ranges through D, the colors $c_i(gx^{-1})$ are all distinct. Thus the coloring is proper.

Now any new color $c_i(g)$ is assigned to the edge from gx to $u_i(x)$ for all x in D. Thus $c_i(g)$ occurs at all nodes of gD. Now any old color occurs at all nodes of some translate hD, which meets gD in at least one node. Thus every new color meets every old color.

Now suppose $c_i(g)$ and $c_j(h)$ are two new colors. By condition (4), select $x, y \in D$ so that $xy^{-1} = g^{-1}h$. Then $gx = hy$. Letting $z = gx$, we then have $zx^{-1} = g$ and $zy^{-1} = h$. Thus the edges $[z, u_i(x)]$ and $[z, u_j(y)]$ are colored $c_i(g)$ and $c_j(h)$, respectively. Hence all new colors also meet one another, so the coloring is complete. □

A well-known theorem of Singer [11] says that if q is a prime power and $n = q^2 + q + 1$, then there is a subset D of Z_n (the cyclic group of order n) such that every nonzero element of Z_n is expressible uniquely as $x - y$ for x, y in D. This difference set arises from the projective plane PG(2, q) over the Galois field GF(q). Of course, D has $q + 1$ elements, so if q is odd, the complete graph on D is 1-factorable and D is an $(n, q + 1, q)$-Bouchet diagram. Thus from Theorem 4.2, we obtain the following corollary.

Corollary 4.3. *Let q be a power of an odd prime and set $n = q^2 + q + 1$. Then for all $k \geqslant 0$, $A(n + k(q+1)) \geqslant qn + kn$.*

5. Summary

An edge coloring of K_n may be represented by a *color table*: an $n \times n$ matrix whose i, j entry is the color on edge $[i, j]$. Of course, such a table is symmetric. The coloring is proper iff each row (and hence each column) contains distinct entries. The coloring is complete iff each pair of colors occurs together in at least one row (column). Color tables for optimal colorings of K_6, K_7, K_8, K_{10}, and K_{11} are given in Fig. 5. (An optimal coloring of K_5 appears in Fig. 2; the $(9, 4, 2)$-Bouchet diagram from Fig. 4 yields an optimal coloring of K_9.)

The values of $A(n)$ for $n \leqslant 7$ were known to Bories [2]. Using differencing methods, Bouchet [3] found $A(8)$ and $A(9)$. Optimal colorings of K_{10} and K_{11} were found by Ray Rowley and Craig Turner, respectively, using ad hoc methods described in Turner et al. [13].

Table 3 shows the currently best upper and lower bounds on $A(n)$ for $n \leqslant 100$. The upper bound is $B(n)$ except as noted:

(a) Theorem 2.5

*	1	2	3	4	5
1	*	5	6	7	3
2	5	*	7	6	1
3	6	7	*	2	4
4	7	6	2	*	8
5	3	1	4	8	*

$n = 6$, $k = 8$
Bories

*	1	2	3	4	5	6
1	*	7	8	6	9	2
2	7	*	5	10	4	8
3	8	5	*	9	1	10
4	6	10	9	*	7	3
5	9	4	1	7	*	11
6	2	8	10	3	11	*

$n = 7$, $k = 11$
Bories

*	1	2	3	4	5	6	7
1	*	5	8	9	10	11	4
2	5	*	6	12	13	8	10
3	8	6	*	7	9	14	12
4	9	12	7	*	1	13	11
5	10	13	9	1	*	2	14
6	11	8	14	13	2	*	3
7	4	10	12	11	14	3	*

$n = 8$, $k = 14$
Bouchet

*	1	2	3	4	5	6	7	8	9	10
1	*	7	11	12	13	14	2	15	16	17
2	7	*	18	19	20	21	1	22	23	24
3	11	18	*	24	16	23	8	6	13	19
4	12	19	24	*	9	15	14	17	5	18
5	13	20	16	9	*	22	4	21	11	23
6	14	21	23	15	22	*	12	20	10	3
7	2	1	8	14	4	12	*	25	26	27
8	15	22	6	17	21	20	25	*	27	26
9	16	23	13	5	11	10	26	27	*	25
10	17	24	19	18	23	3	27	26	25	*

$n = 11$, $k = 27$
Turner

*	1	2	3	4	5	6	7	8	9
1	*	5	10	11	12	13	14	15	16
2	5	*	16	8	17	18	19	20	21
3	10	16	*	6	19	4	17	13	22
4	11	8	6	*	22	3	20	14	12
5	12	17	19	22	*	15	1	9	11
6	13	18	4	3	15	*	21	10	7
7	14	19	17	20	1	21	*	5	18
8	15	20	13	14	9	10	5	*	2
9	16	21	22	12	11	7	18	2	*

$n = 10$, $k = 22$
Rowley

Fig. 5. Color tables of optimal k-colorings of $L(K_n)$.

Table 3. Best current bounds on $A(n)$ for $1 \leqslant n \leqslant 100$. Lower bounds arising from $A(n+1) \geqslant A(n)$ are omitted.

n	1	2	3	4	5	6	7	8	9	10
Upper	—	1	3	3	7	8	11	14	18	22
Lower	—	1	3	3	7	8	11	14	18	22
						b, c	*c*	*c*	*d*	*c*
n	11	12	13	14	15	16	17	18	19	20
Upper	27	33	39	44	49	53	57	61	65	69
Lower	27	31	39		41		52		57	
	c	*s*	*p*	*a*	*m*		*e*		*d*	
n	21	22	23	24	25	26	27	28	29	30
Upper	73	77	84	92	100	108	117	126	135	145
Lower	65		83	89	100	105	110		112	136
	e		*p*	*s*	*d*	*p*	*p*		*m*	*s*
n	31	32	33	34	35	36	37	38	39	40
Upper	155	165	174	181	187	193	199	205	211	217
Lower	155		157		159		186		188	
	p		*a, m*		*m*		*e*		*m*	
n	41	42	43	44	45	46	47	48	49	50
Upper	223	229	235	241	247	258	270	282	294	306
Lower	190		219		221		223		250	
	m		*e*		*m*		*m*		*e*	
n	51	52	53	54	55	56	57	58	59	60
Upper	318	331	344	357	371	385	399	413	427	440
Lower	252		254		288	343	399		401	
	m		*m*		*t*	*t*	*p*		*m*	*a*
n	61	62	63	64	65	66	67	68	69	70
Upper	449	457	465	473	481	489	497	505	513	521
Lower	403		405		456		458		460	
	m		*m*		*e*		*m*		*m*	
n	71	72	73	74	75	76	77	78	79	80
Upper	529	539	545	553	561	570	585	600	616	632
Lower	462		513		515		583	591		593
	m		*e*		*m*		*p*	*s*		*m*
n	81	82	83	84	85	86	87	88	89	90
Upper	648	664	680	697	714	731	748	765	783	801
Lower		657	666		668		670		672	728
		p	9		*m*		*m*		*m*	*t*
n	91	92	93	94	95	96	97	98	99	100
Upper	819	837	855	874	890	901	911	921	931	941
Lower	819		821		823		825		827	
	p		*m*		*a, m*		*m*		*m*	

(b) Bories–Turner Theorem 2.7

The lower bounds come from these sources:

(c) Color tables in Fig. 5
(d) Bouchet diagrams in Fig. 4
(e) Extension of Bouchet diagrams in Corollary 4.3
(m) Monotonicity Theorem 1.6 ($A(n+2) \geq A(n) + 2$)
(p) Projective planes (Theorems 1.2, 3.1, and 3.2)
(s) Special constructions from Section 3
(t) The trivial bound $A(n) \geq A(n+1) - n$

Lower bounds which arise simply from the monotonicity of $A(n)$ are omitted from the table.

Note added in proof

Improvements on some of the lower bounds in Table 3 using a modification of Bouchet diagrams have come to light since submission of this paper. In particular, $A(12) \geq 32$ and improvements for $46 \leq n \leq 49$ are known. Details will be reported elsewhere.

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